TEMPERATURE RELATIONS INSIDE BODIES IN NONLINEAR HEAT-CONDUCTION PROCESSES

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An equation has been obtained which allows the unsteady temperature field inside multidimensional bodies to be calculated from known temperatures along the coordinate axes, in nonlinear heat-condition processes.

It is well known that in the majority of practical problems (especially under conditions of intense heating), the heat flux at the surface of a body is associated with the surface temperature by the nonlinear relation

$$(\operatorname{grad}\theta)_{surf} = f(\theta_{surf}), \tag{1}$$

or, using the form of boundary conditions of the third kind,

$$(\operatorname{grad}\theta)_{surf} = \mathrm{Bi}(\theta_{surf})(1-\theta_{surf}).$$

A typical example of such high-temperature heating is radiative heat exchange at a surface, when the boundary condition is given by the Stefan-Boltzmann law

$$(\operatorname{grad}\theta)_{surf} = \mathrm{Sk}(1-\theta^4_{surf}). \tag{2}$$

The analytical study of such processes presents great mathematical difficulty, and therefore, numerical methods of solution with high-speed computers [1–3] are used in the majority of cases.

Interesting attempts at theoretical investigation of unsteady heat conduction with various nonlinear boundary conditions have been made by Biot [4], using the variational principle. Calculation of the temperature field with nonlinear integral equations has been performed in [5, 6].

Reference [7] describes a simple and reliable method of computing unsteady heat conduction with boundary conditions of type 1. Using this method we have been able to establish the relation, important for technical applications, between the temperatures inside multidimensional bodies and the temperature distribution along the coordinate axes, for a number of special cases of the functional dependence (1).

Let us demonstrate the derivation of the equation in an example of heating of a two-dimensional body.

Following the method of [7], we can write the system of differential equations describing the unsteady temperature field in the rectangular region $2R_x \times 2R_y$ as

$$\frac{1}{a}\,\frac{\partial\theta(x, y, \tau)}{\partial\tau} = \nabla^2\theta(x, y, \tau), \tag{3}$$

$$\frac{\partial\theta(R_x, y, \tau)}{\partial x} = Kf[\theta(R_x, y, \tau)], \tag{4}$$

$$\frac{\partial\theta(x, R_y, \tau)}{\partial y} = Kf[\theta(x, R_y, \tau)], \tag{5}$$

$$\frac{\partial\theta(0, y, \tau)}{\partial x} = \frac{\partial\theta(x, 0, \tau)}{\partial y} = 0, \tag{6}$$

$$\theta(x, y, 0) = \theta_{init}, \tag{7}$$

which may be transformed by means of the function

$$W(x, y, \tau) = \exp\left\{-p\int\limits_0^0 \frac{d\theta(x, y, \tau)}{f[\theta(x, y, \tau)]}\right\}, \tag{8}$$

where p is a parameter, to the form

$$\frac{1}{a}\,\frac{\partial W(x, y, \tau)}{\partial\tau} = \nabla^2 W(x, y, \tau) - \varphi(x, y, \tau), \tag{9}$$

$$\varphi(x, y, \tau) = pW(x, y, \tau)\times$$
$$\times\frac{\left(\frac{\partial\theta}{\partial x}\right)^2 + \left(\frac{\partial\theta}{\partial y}\right)^2}{[f(\theta)]^2}\left[p + \frac{df(\theta)}{d\theta}\right], \tag{10}$$

$$\frac{\partial W(R_x, y, \tau)}{\partial x} = -pKW(R_x, y, \tau), \tag{11}$$

$$\frac{\partial W(x, R_y, \tau)}{\partial y} = -pKW(x, R_y, \tau), \tag{12}$$

$$\frac{\partial W(0, y, \tau)}{\partial x} = \frac{\partial W(x, 0, \tau)}{\partial y} = 0, \tag{13}$$

$$W(x, y, 0) = \exp\left[-p\int\limits_0^{\theta_{surf}}\frac{d\theta}{f(\theta)}\right] = W_{surf}. \tag{14}$$

Here $\theta = T/T_m$, where T_m is the temperature of the heating medium, $f(\theta)$ is a nonlinear function of temperature θ; K is a constant.

The purpose of this transformation is to linearize boundary conditions (4)–(5) and reduce them to the ordinary linear form of boundary conditions of the third kind. Then the nonlinear complex in (10) in the transformed heat-conduction equation (9) can be reduced to the required minimum by appropriate choice of the parameter p. Recommendations are made in [7] regarding the choice of parameter p when the nonlinear boundary condition (1) is given in general form; for the particular cases of radiative heating (2), and also for radiative-convective heating, the determination of p is given in [8, 9].

Now, using the solution of problem (9)–(14) for $\varphi(x, y, \tau) = 0$ [10], and transformation (8), we find the temperature field $\theta = \theta(x, y, \tau)$.

In accordance with the well-known property of the standard form (11), the temperature $W(x, y, \tau)$ with

Comparison of Values of Relative Temperature at the Center of an Infinite Prism, Obtained by Various Methods

Fo	Data for computer calculation				According to (18)	Error δ, per cent
	θ (0; 0; 708; Fo)	θ(0; 630; 0; Fo)	θ (0.630; 0.708; Fo)	θ (0; 0; Fo)	θ (0; 0; Fo)	
0.10	0.5627	0.4589	0.7108	0.1767	0.2191	24.00
0.20	0.6943	0.6172	0.8284	0.3320	0.3557	7.14
0.30	0.7819	0.7249	0.8831	0.4982	0.5100	2.37
0.40	0.8435	0.8016	0.9172	0.6315	0.6378	1.00
0.50	0.8911	0.8612	0.9408	0.7317	0.7323	0.08
0.75	0.9506	0.9367	0.9741	0.8801	0.8806	0.06
1.00	0.9783	0.9722	0.9886	0.9471	0.9473	0.02
1.25	0.9904	0.9877	0.9950	0.9766	0.9764	0.02
1.50	0.9958	0.9946	0.9978	0.9897	0.9897	0.00
2.00	0.9992	0.9989	0.9996	0.9980	0.9980	0.00

$\varphi(x, y, \tau) = 0$, in the regular temperature regime situation, can be expressed in the form

$$W(x, y, \tau) = X(x) \cdot Y(y) \cdot T(\tau).$$

For our case

$$W(x, y, \tau) = \mathrm{const} \cdot \cos \mu_x \frac{x}{R_x} \cdot \cos \mu_y \frac{y}{R_y} \times$$
$$\times \exp\left[-(\mu_x^2 K_x^2 + \mu_y^2 K_y^2) \frac{a\tau}{R^2}\right].$$

Then

$$X(x) \cdot Y(y) \cdot T(\tau) = \exp\left[-p \int_0^{\theta(x,y,\tau)} \frac{d\theta}{f(\theta)}\right]. \qquad (15)$$

Now, by giving certain fixed values to the coordinates $x = x_0$, and then $y = y_0$, we obtain

$$Y(y) = \frac{\exp\left[-p \int_0^{\theta(x_0,y,\tau)} \frac{d\theta}{f(\theta)}\right]}{X(x_0) \cdot T(\tau)}, \qquad (16)$$

$$X(x) = \frac{\exp\left[-p \int_0^{\theta(x,y_0,\tau)} \frac{d\theta}{f(\theta)}\right]}{Y(y_0) \cdot T(\tau)}. \qquad (17)$$

After substituting X (x) and Y (y) into (15) we obtain a general relationship between the relative temperatures inside the body and the temperature distribution along the coordinates x_0, y_0:

$$\int_{\theta(x_0,y,\tau)}^{\theta(x,y,\tau)} \frac{d\theta}{f(\theta)} = \int_{\theta(x_0,y_0,\tau)}^{\theta(x,y_0,\tau)} \frac{d\theta}{f(\theta)}. \qquad (18)$$

The dependence for three-dimensional bodies is obtained analogously.

For the special case of radiative heating (boundary condition (2)), Eq. (18) takes the form

$$\mathrm{Arth} \frac{\theta(x, y, \tau) + \theta(x_0, y_0, \tau)}{1 + \theta(x, y, \tau)\,\theta(x_0, y_0, \tau)} +$$
$$+ \mathrm{arctg} \frac{\theta(x, y, \tau) + \theta(x_0, y_0, \tau)}{1 - \theta(x, y, \tau)\,\theta(x_0, y_0, \tau)} =$$
$$= \mathrm{Arth} \frac{\theta(x_0, y, \tau) + \theta(x, y_0, \tau)}{1 + \theta(x_0, y, \tau)\,\theta(x, y_0, \tau)} +$$
$$+ \mathrm{arctg} \frac{\theta(x_0, y, \tau) + \theta(x, y_0, \tau)}{1 - \theta(x_0, y, \tau)\,\theta(x, y_0, \tau)}. \qquad (19)$$

Our calculations and analysis of other experimental data confirm the theoretical relationships (18) and (19).

As an example we examine heating of a prism-shaped block of square section 2R × 2R, with initial temperature $\theta_{init} = 0.15$. The nonlinear boundary condition (1) is given by the expression

$$(\mathrm{grad}\,\theta)_{surf} = \mathrm{Bi}_0 \mathrm{Bi}(\theta_{surf})(1 - \theta_{surf}) =$$
$$= 2(1 + 0.5\theta_{surf})(1 - \theta_{surf}).$$

The table compares values of relative temperatures at the center of an infinite prism $\theta(0, 0, \mathrm{Fo})$, as obtained on the "Minsk-1" computer, to those found on the basis of Eq. (18). The relative coordinates x_0/R and y_0/R were 0.630 and 0.708, respectively.

As can be seen from the table, with increase of Fo the central point of the infinite prism gradually reaches a regular thermal regime, and the temperature variation there begins to conform to Eq. (18).

Equation (18) allows us to reconstruct the unsteady temperature field within multidimensional bodies from readings of thermocouples along the coordinate axes. The origin of coordinates can be located at any point inside the body.

It should be noted in particular, that when using relationships of the type (19), obtained from (18), there is no need to know the physical parameters of the material: thermal conductivity, density, specific heat, and surface emissivity.

Relation (18) is strictly valid for a regular thermal regime, and can be used in investigations of heat propagation processes.

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